$$x^2 + 3x + 2 \overline{\smash{\big)}\ x^2 + 6}$$
$$\underline{-(x^2 + 3x + 2)}$$
$$-3x + 4$$

With quotient 1 above.

Entry Task: Evaluate

$$\int \frac{x^2 + 6}{x^2 + 3x + 2}\, dx$$

$$= \int 1 + \frac{-3x + 4}{x^2 + 3x + 2}\, dx$$

$$= \int 1 + \frac{7}{x+1} - \frac{10}{x+2}$$

$$= \boxed{x + 7\ln|x+1| - 10\ln|x+2| + C}$$

$$\frac{-3x + 4}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$-3x + 4 = A(x+2) + B(x+1)$$

$$x = -1 \Rightarrow 7 = A$$

$$x = -2 \Rightarrow 10 = B(-1) \qquad B = -10$$

CHECK : $\quad -3x + 4 = (A+B)x + (2A+B)$

$$A + B = -3 \ \checkmark$$
$$2A + B = 4 \ \checkmark$$

CHECK:

$$1 + \frac{7}{x+1} - \frac{10}{x+2}$$

Simplify or plug in values

$$x = 0 \qquad 1 + 7 - \frac{10}{2} = 7 \qquad \frac{0 + 6}{0 + 0 + 2} = 7$$

Example: Evaluate

$$\int \frac{x^2 - x + 6}{x^3 + 3x} \, dx$$

$$\frac{x^2 - x + 6}{x(x^2 + 3)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 3}$$

$$\Rightarrow \quad x^2 - x + 6 = A(x^2 + 3) + (Bx + c)x$$
$$= Ax^2 + 3A + Bx^2 + Cx$$
$$x^2 - x + 6 = (A + B)x^2 + Cx + 3A$$

$$\Rightarrow \quad A + B = 1 \quad \Rightarrow B = 1 - A$$
$$C = -1$$
$$3A = 6 \Rightarrow A = 2$$
$$B = 1 - A \Rightarrow B = -1$$

$$\int \frac{2}{x} + \frac{-x - 1}{x^2 + 3} \, dx$$

$$= \int \frac{2}{x} + \frac{-x}{x^2 + 3} + \frac{-1}{x^2 + 3} \, dx$$

$$= 2\ln|x| - \int \frac{x}{x^2 + 3} \, dx + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$$

$$\int \frac{x}{u} \cdot \frac{1}{2x} \, du$$

$$\frac{1}{2} \ln|u|$$

$$u = x^2 + 3$$
$$du = 2x \, dx$$
$$\frac{1}{2x} du = dx$$

$$= \boxed{2\ln|x| - \frac{1}{2}\ln|x^2 + 3| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c}$$

Example:

$$\int \frac{x}{x^2 + 4x + 5} dx$$

IRREDUCIBLE!

$x^2 + 4x + 5 \overset{?}{=} 0$ HAS NO REAL SOL'NS

$$x = \frac{-4 \pm \sqrt{16 - 4(5)}}{2}$$

-4

$x^2 + 4x + 4 - 4 + 5$

$(x+2)^2 + 1$ CHECK!

$$\int \frac{x}{(x+2)^2 + 1} dx$$

$t = x + 2$ $x = t - 2$

$dt = dx$

$$\int \frac{t - 2}{t^2 + 1} dt$$

$$\int \frac{t}{t^2 + 1} dt - \int \frac{2}{t^2 + 1} dt$$

$u = t^2 + 1$

$du = 2t \, dt$

$\frac{1}{2t} du = dt$

$$\int \frac{t}{u} \frac{1}{2t} du - 2 \tan^{-1}(t) + C$$

$$\frac{1}{2} \ln |u|$$

$$= \frac{1}{2} \ln |t^2 + 1| - 2 \tan^{-1}(t) + C$$

$$= \boxed{\frac{1}{2} \ln ((x+2)^2 + 1) - 2 \tan^{-1}(x+2) + C}$$

How to integrate

1. Look for simplifications/substitutions
3. Products/Logs/Inverse Trig →BY PARTS
   Sin/Cos/Tan/Sec combos   → TRIG
   Quadratic (under a radical) →TRIG SUB
   Rational Function   →PART. FRAC.
2. If nothing seems to work, substitution.
   (u = inside, u = $\sqrt{}$, u = trig, u = e^x)

Examples:

1. $\int e^{\sqrt{x}}dx$

$t=\sqrt{x} \implies t^2 = x$

$2t\,dt = dx$

$\int e^t \, 2t\,dt$

$\int 2t \, e^t dt$

$u = 2t \qquad dv = e^t dt$

$du = 2dt \qquad v = e^t$

$= 2t e^t - \int 2e^t dt$

$= 2t e^t - 2 e^t + c$

$= \boxed{2\sqrt{x} \, e^{\sqrt{x}} - 2 e^{\sqrt{x}} + c}$

2. $\displaystyle\int \frac{\cos(x)}{4-\sin^2(x)}\,dx$

$t = \sin(x)$

$dt = \cos(x)\,dx$

$\displaystyle\int \frac{1}{4-t^2}\,dt$

$\dfrac{1}{(2-t)(2+t)} = \dfrac{A}{2-t} + \dfrac{B}{2+t}$

$1 = A(2+t) + B(2-t)$

$t = 2 \implies A = \frac{1}{4}$

$t = -2 \implies B = \frac{1}{4}$

$= \displaystyle\int \frac{1/4}{2-t} + \frac{1/4}{2+t}\,dt$

$= \left(-\frac{1}{4}\ln|2-t| + \frac{1}{4}\ln|2+t| + c \right)$

$\uparrow$

DON'T
FORGET

$= -\frac{1}{4}\ln|2-\sin(x)| + \frac{1}{4}\ln|2+\sin(x)| + c$

3. $\displaystyle\int \frac{3}{x-2\sqrt{x}}\,dx$

$t = \sqrt{x}$

$t^2 = x$

$2t\,dt = dx$

$\displaystyle\int \frac{3}{t^2-2t}\,2t\,dt$

$\displaystyle\int \frac{6t}{t(t-2)}\,dt$

$= 6\ln|t-2| + c$

$= \boxed{6\ln|\sqrt{x}-2| + c}$

1. $\int e^x \cos(e^x) \sin^3(e^x) \, dx$

$t = e^x$

$dt = e^x dx$

$\int \cos(t) \sin^3(t) \, dt$

$u = \sin(t)$

$du = \cos(t) dt$

$\int u^3 \, du$

$\frac{1}{4} u^4 + c$

$= \frac{1}{4} \sin^4(e^x) + c$
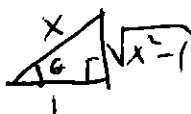
How would you _start_ these?

1. $\int \tan^3(x)\sec(x)\,dx = \int \tan^2(x)\sec(x)\tan(x)\,dx$ $u = \sec(x)$

    TRIG! $= \int (\sec^2(x) - 1)\sec(x)\tan(x)\,dx$ $du = \sec(x)\tan(x)\,dx$
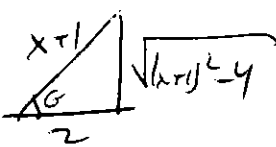
2. $\int x^2 \ln(x)\,dx$ $= \frac{1}{3}x^3 \ln(x) - \int \frac{1}{3}x^2\,dx$ $u = \ln(x)$ $dv = x^2\,dx$

    BY PARTS! $= \frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3 + C$ $du = \frac{1}{x}\,dx$ $v = \frac{1}{3}x^3$

3. $\int x\sqrt{5 - x^2}\,dx = \int x\sqrt{u}\,\frac{1}{-2x}\,du = -\frac{1}{2}\frac{2}{3}u^{3/2} + C$ $u = 5 - x^2$ ← EASIER $x = \sqrt{5}\sin(\theta)$

    SUB. (OR TRIG SUB) $= -\frac{1}{3}(5 - x^2)^{3/2} + C$ $du = -2x\,dx$ OR

                                                 $-\frac{1}{2x}\,du = dx$

4. $\int \frac{\sqrt{x^2 - 1}}{x^2}\,dx = \int \frac{\sqrt{\sec^2\theta - 1}}{\sec^2\theta}\sec\theta\tan\theta\,d\theta = \int \frac{\tan^2\theta}{\sec\theta}\,d\theta$ OR $x = \sec\theta$

    TRIG SUB. $\int \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta = \int \frac{\sin^2\theta}{\cos\theta}\,d\theta$ $dx = \sec\theta\tan\theta\,d\theta$

5. $\int \frac{x^2 + 1}{x^2 - 2x - 3}\,dx = \int 1 + \frac{2x + 4}{x^2 - 2x - 3}\,dx$ $x^2 - 2x - 3\,)\overline{x^2 + 1}$

    DIVIDE, THEN FACTOR DENOM $\frac{2x + 4}{(x - 3)(x + 1)} = \frac{A}{x - 3} + \frac{B}{x + 1}$ $\frac{-(x^2 - 2x - 3)}{2x + 4}$

6. $\int x\tan^{-1}(x)\,dx = \frac{1}{2}x^2\tan^{-1}(x) - \frac{1}{2}\int \frac{x^2}{x^2 + 1}\,dx$ $u = \tan^{-1}(x)$ $dv = x\,dx$

    BY PARTS   ↖ DIVIDE!! $du = \frac{1}{x^2 + 1}\,dx$ $v = \frac{1}{2}x^2$

7. $\int \frac{dx}{\sqrt{4x^2 + 8x - 12}}\,dx$ $\sqrt{4(x^2 + 2x + 1 - 1 - 3)} = 2\sqrt{(x + 1)^2 - 4}$

                ↓ $\int \frac{1}{2\sqrt{4\sec^2\theta - 4}}\,2\sec\theta\tan\theta\,d\theta$ $x + 1 = 2\sec\theta$

    COMPLETE SQUARE

    & TRIG. SUB! $= \frac{1}{2}\int \frac{1}{2\tan\theta}\,2\sec\theta\tan\theta\,d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$